$$\frac{1}{ze^{-1/z}} = \frac{1}{z}e^{-1/z} = \left(\frac{1}{z}\right)(\cdots\cdots)$$

Thm (Riemann Removable Singularity Thm.)

Let z_0 be an isolated singularity of a holomorphic fcn f. Then the following are equivalent:

① z_0 is a removable singularity of f

② $f(z)$ is bounded on some disk centered at z_0. That is, $\exists \varepsilon > 0$ s.t. $f\big|_{B(z_0,\varepsilon)}$ is bounded.

③ $\lim_{z\to z_0} f(z)$ exists (and equals the constant term in the Laurent series).

In this case, the fcn g defined by
$$g(z) = \begin{cases} f(z) & z \neq z_0 \\ \lim_{z\to z_0} f(z) & z = z_0 \end{cases}$$
is holom. (on $B(z_0,\varepsilon)$) near z_0.

—

② Is very useful: if we have a holom. fcn. that is bounded near an isolated singularity, then we know the singularity is removable.

② Characterization of pole singularities

If z_0 is a pole of order m for the holomorphic fcn f,

then $f(z) = a_{-m}(z-z_0)^{-m} + a_{-m+1}(z-z_0)^{-m+1} + \dots$

is the Laurent series of f on a deleted neighborhood $B^*(z_0, \varepsilon)$.

Then

$$g(z) = (z-z_0)^m f(z)$$

$$= a_{-m} + a_{-m+1}(z-z_0)^1 + \dots$$

has a removable singularity at z_0,

so can be extended to be holomorphic on $B(z_0, \varepsilon)$.

holomorphic.

$$\Rightarrow f(z) = \frac{g(z)}{(z-z_0)^m}$$

Another way to get poles:

let α, β be any two holomorphic fcns defined on a domain D.

(β is not allowed to be the zero fcn)

Then let $f(z) = \dfrac{\alpha(z)}{\beta(z)}$.

Then $f(z)$ might have singularities, but they are all poles or removable singularities.

Functions defined like this are

called <u>meromorphic fcns on D.</u>

Why can't essential singularities occur? $\forall z_0 \in D$, it's possible that $\beta(z_0) = 0$, But it must be a zero of some order k (otherwise β would be identically zero if it was an ∞ order zero). $\therefore$ $\beta(z) = (z-z_0)^k \gamma(z)$, where $\gamma(z_0) \neq 0$, so $\gamma(z) \neq 0$ in a nbhd of z_0.

$$\therefore \quad \frac{\alpha(z)}{\beta(z)} = \frac{\alpha(z) \cdot (z-z_0)^{-k}}{\gamma(z)}$$

holomorphic near z_0

$$= \underbrace{\left(a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\right)(z-z_0)^{-k}}_{\text{can only have a finite \# of neg terms}}$$

- If z_0 is a pole of order m for $f(z)$, then

$$\lim_{z \to z_0} f(z) = \infty.$$

$$f(z) = a_{-m}(z-z_0)^{-m} + \cdots$$

$$= (z-z_0)^{-m} \cdot [g(z)]$$

holomorphic, nonzero near z_0.

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \underbrace{|z-z_0|^{-m}}_{\to +\infty} \underbrace{|g(z)|}_{\to |g(z_0)| \neq 0}$$

$$= \infty.$$

[We mean $\lim_{z \to z_0} |f(z)| = \infty$ when we say $\lim_{z \to z_0} f(z) = \infty$)

③ Characterization of essential singularities.

Let z_0 be an essential singularity of f

- $\lim_{z \to z_0} f(z)$ does not exist & does not $= \infty$.
- for every $\varepsilon > 0$

$$f(B(z_0,\varepsilon)) = \begin{cases} \mathbb{C} \\ \text{or} \\ \mathbb{C} \setminus \{p\} \end{cases}$$

for some fixed $p \in \mathbb{C}$. indep. of ε.

This says: for every ε-nbhd of z_0, the values of the function are every possible point in $\mathbb{C}$ (with the possible excepting one complex #)

↑ I think this is called "Picard's theorem".

So essential singularities are totally wacky!

examples.

Essential singularities: put $\left(\frac{1}{z-z_0}\right)$ into a Taylor series you like for an entire fcn.

eg. $f(z) = e^{\frac{1}{z-4}}$ has essential singularity at $z=4$.

$f(B_\varepsilon(4)) = \mathbb{C} \setminus \{0\}$

↑ exponential fcns are never zero.

• $g(z) = \sin\left(\frac{1}{z-4}\right)$ has an essential singularity at $z=4$,

& $g(B_\epsilon(4)) = \mathbb{C}$.

$\Longleftrightarrow \forall w \in \mathbb{C}, \forall \epsilon > 0, \exists z \in B_\epsilon(4)$ s.t. $\sin\left(\frac{1}{z-4}\right) = w$.

actually, there's an infinite # of solutions!

__Example__: Classify all the singularities of

$$g(z) = \frac{\left(e^{z}-1\right)^{2} \cos\left(\frac{2}{z+\pi}\right)}{(z+2)\left(\sin(2z)\right)^{3}}$$

__Solution:__

- Where are the singularities?

$z=-2$, $z=\frac{k\pi}{2}$ for $k \in \mathbb{Z}$.

For $z=-2$, $g(z) = (z+2)^{-1} \cdot \underbrace{\overbrace{\left[\frac{\left(e^{z}-1\right)^{2}\cos\left(\frac{2}{z+\pi}\right)}{\left(\sin(2z)\right)^{3}}\right]}^{\text{bubba}}}_{\text{holom at } z=-2}$

$$g(z) = (z+2)^{-1}\left[a_0 + a_1(z+2) + a_2(z+2)^2 + \dots\right]$$

$a_0 \neq 0$, because bubba$(-2) = a_0$,

$$\text{bubba}(-2) = \frac{\left(e^{-2}-1\right)^{2}\cos\left(\frac{2}{-2+\pi}\right)}{\left(\sin(2(-2))\right)^{3}} \neq 0.$$

$\therefore$ $z=-2$ is a pole of order 1.

Now consider the singularities $z=\frac{k\pi}{2}$ for $k \neq -2$.

$$g(z) = \frac{1}{(\sin(2z))^3} \cdot \frac{(e^z-1)^2 \cos\left(\frac{2}{z+\pi}\right)}{z+2}$$

buffer a_z — holomorphic at $z=\frac{k\pi}{2}$

$$g(z) = \frac{1}{(\sin(2z))^3} \cdot \left[a_0 + a_1\left(z-\frac{k\pi}{2}\right) + a_2\left(z-\frac{k\pi}{2}\right)^2 + \dots\right]$$

$\neq 0$ because

$$a_0 = \frac{\left(e^{k\pi/2}-1\right)^2 \cos\left(\frac{2}{\frac{k\pi}{2}+\pi}\right)}{\frac{k\pi}{2}+2} \neq 0$$

Side calculation

$$(\sin(2z))^3 = \left(b_0 + b_1\left(z-\frac{k\pi}{2}\right) + b_2\left(z-\frac{k\pi}{2}\right)^2 + \dots\right)^3$$

$z=\frac{k\pi}{2}$, $b_0 = 0$

$$b_0 = \sin\left(2\left(\frac{k\pi}{2}\right)\right) = 0$$

$$b_1 = 2\cos(2z)\Big|_{z=\frac{k\pi}{2}} = 2\cos(k\pi) = 2(-1)^k \neq 0.$$

$$(\sin(2z))^3 = \left(z - \frac{k\pi}{2}\right)^3 \left[\underbrace{\cancel{b_1}_{0} + b_2\left(z-\frac{k\pi}{2}\right) + b_3\left(z-\frac{k\pi}{2}\right)^2 + \dots}_{\text{babba}}\right]^3$$

babba $\Rightarrow$ holomorphic nonzero at $z = k\pi/2$

$$\Rightarrow g(z) = \frac{1}{(\sin(2z))^3} \cdot \left[\cancel{a_0}^{0} + a_1\left(z-\frac{k\pi}{2}\right) + \dots\right]$$

$$g(z) = \left(z - \frac{k\pi}{2}\right)^{-3} \cdot \left[\begin{matrix}\text{non zero}\\ \text{holom}\end{matrix}\right]$$

$$\Rightarrow \left[c_0 + c_1\left(z-\frac{k\pi}{2}\right) + \dots\right]$$

$\therefore z = \frac{k\pi}{2}$ is a pole of order 3 if $k \neq -2$.

$\underline{z = -\pi}$

$$g(z) = \left(\begin{matrix}\text{something}\\ \text{not always } 0\end{matrix}\right) \underbrace{\cos\left(\frac{z}{z+\pi}\right)}_{\text{essential sing.}}$$

$\Rightarrow \infty$ # of neg terms.

$\Rightarrow z = -\pi$ is an essential singular

$$\frac{\sin(z)}{z} \to 1$$

$$\frac{\sin(z)}{z} = \left(c_0 + c_1 z + \cdots\cdots\right)$$